Discrete Anger functions

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Abstract

abcd

1 Introduction

We shall investigate the solution of the inhomogeneous second-order difference equation

$$t(t-1)\Delta^2 y(t-2) + t\Delta y(t-1) + t(t-1)y(t-2) - \nu^2 y(t) = \frac{(t-\nu)\sin(\pi\nu)}{\pi}, \quad (1)$$?angerdifferenceeqt?

which we call the discrete Anger difference equation.

The following inhomogeneous second-order differential equation is known as the Anger differential equation [2]:

$$z^2 y'' + zy' + (z^2 - \nu^2)y = \frac{(z-\nu)\sin(\pi\nu)}{\pi}, \quad (2)$$?classicalangerde?

and has....

classical bessel...applications... The discrete Bessel functions of the first kind, J_ν, were introduced in [1] as solutions of a certain second-order homogeneous equation called the Bessel difference equation. Numerous recent works have used the J_ν to solve discrete wave equation IVPs [4, Theorem 3.1], pursued the "modified" discrete Bessel functions I_ν in solving semidiscrete diffusion problems [5, Lemma 2.2], usage of the discrete time heat kernel in field theories of physics [3], and the J_ν have been generalized to a broader class via the time scales calculus [6].

Our task is to consider a discrete Anger difference equation analogous to (**??**).

2 Preliminaries and definitions

For $n \in \{0, 1, 2, \ldots\}$, the Pochhammer symbol $(a)_n$ is defined by

$$(a)_n = a(a+1)...(a+n-1). \quad (3)$$?pochhammer?

The classical generalized hypergeometric series ${}_p\mathcal{F}_q$ is defined by the formula

$$ {}_p\mathcal{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;t)=\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k t^k}{(b_1)_k\ldots(b_q)_k k!}. \tag{4} $$?genhyper?

The classical Anger hypergeometric series is defined by

$$ \mathfrak{J}_\nu(z)=\frac{1}{\Gamma(1-\frac{\nu}{2})\Gamma(1+\frac{\nu}{2})}\cos\frac{\pi\nu}{2}\,{}_1\mathcal{F}_2\left(1;1-\frac{1}{2}\nu,1+\frac{1}{2}\nu;-\frac{1}{4}z^2\right) + \frac{z}{2\Gamma(\frac{3-\nu}{2})\Gamma(\frac{3+\nu}{2})}\sin\frac{\pi\nu}{2}\,{}_1\mathcal{F}_2\left(1;\frac{1}{2}(3-\nu),\frac{1}{2}(3+\nu);-\frac{1}{4}z^2\right). \tag{5} $$?classhyper?

Equation (5) solves (**??**). The classical Anger power series is defined as

$$ \mathfrak{J}_\nu(z)=\cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k\Gamma(k+\frac{\nu}{2}+1)\Gamma(k-\frac{\nu}{2}+1)}z^{2k} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k+1}\Gamma(k+\frac{\nu}{2}+\frac{3}{2})\Gamma(k-\frac{\nu}{2}+\frac{3}{2})}z^{2k+1}. \tag{6} $$ {?}

The falling powers are defined as the polynomial

$$ a^{\underline{n}}=a(a-1)(a-2)\ldots(a-n+1). \tag{7} $$ {?}

Known Anger function recurrence relations are

$$ z\mathfrak{J}_{\nu-1}(z)+z\mathfrak{J}_{\nu+1}(z)=2\nu\mathfrak{J}_\nu(z)-\frac{2\sin(\pi\nu)}{\pi} \tag{8} $$?rec1?

$$ \mathfrak{J}_{\nu-1}(z)-\mathfrak{J}_{\nu+1}(z)=2\frac{\delta}{\delta z}\mathfrak{J}_\nu(z) \tag{9} $$?rec2?

$$ z\frac{\delta}{\delta z}\mathfrak{J}_\nu(z)\pm\nu\mathfrak{J}_\nu(z)=\pm z\mathfrak{J}_{\nu\mp 1}(z)\pm\frac{2\sin\pi\nu}{\pi}. \tag{10} $$?rec3?

3 The Anger Difference Equation

We define the discrete Anger function $\mathbf{J}_\nu$ by

$$ \mathbf{J}_\nu(t)=\cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{\underline{2k}}}{2^{2k}\Gamma(k+\frac{\nu}{2}+1)\Gamma(k-\frac{\nu}{2}+1)} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{\underline{2k+1}}}{2^{2k+1}\Gamma(k+\frac{\nu}{2}+\frac{3}{2})\Gamma(k-\frac{\nu}{2}+\frac{3}{2})} \tag{11} $$?angerfunct?

and the discrete Anger hypergeometric be given by

$$ \mathbf{J}_\nu(t)=\frac{1}{\Gamma(1-\frac{\nu}{2})\Gamma(1+\frac{\nu}{2})}\cos\left(\frac{\pi\nu}{2}\right){}_1F_2\left(1;1-\frac{1}{2}\nu,1+\frac{1}{2}\nu;t,2,-\frac{1}{4}\right) + \frac{t}{2\Gamma(\frac{3-\nu}{2})\Gamma(\frac{3+\nu}{2})}\sin\left(\frac{\pi\nu}{2}\right){}_1F_2\left(1;\frac{1}{2}(3-\nu),\frac{1}{2}(3+\nu);t-1,2,-\frac{1}{4}\right). \tag{12} $$ {?}

Theorem 1. *The function* $y(t) = \boldsymbol{J}_\nu(t)$ *satisfies* (1).

Proof. Define the following notations:

$$\alpha_k = \frac{(-1)^k}{4^k\Gamma(k+\frac{\nu}{2}+1)\Gamma(k-\frac{\nu}{2}+1)}, \quad \beta_k = \frac{(-1)^k}{2^{2k+1}\Gamma(k+\frac{\nu}{2}+\frac{3}{2})\Gamma(k-\frac{\nu}{2}+\frac{3}{2})}.$$

Thus we express $\mathbf{J}_\nu(t) = \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^\infty \alpha_k t^{\underline{2k}} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^\infty \beta_k t^{\underline{2k+1}}$. Now we compute

$$\Delta\mathbf{J}_\nu(t) = \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^\infty (2k)\alpha_k t^{\underline{2k-1}} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^\infty (2k+1)\beta_k t^{\underline{2k}},$$

and

$$\Delta^2\mathbf{J}_\nu(t) = \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^\infty (2k)(2k-1)\alpha_k t^{\underline{2k-2}} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^\infty (2k+1)(2k)\beta_k t^{\underline{2k-1}}.$$

Thus,

$$\begin{aligned}
\mathbf{J}_\nu(t-2) &= \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^\infty \alpha_k (t-2)^{\underline{2k}} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^\infty \beta_k (t-2)^{\underline{2k+1}}, \\
\Delta\mathbf{J}_\nu(t-1) &= \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^\infty (2k)\alpha_k (t-1)^{\underline{2k-1}} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^\infty (2k+1)\beta_k (t-1)^{\underline{2k}}, \\
\Delta^2\mathbf{J}_\nu(t-2) &= \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^\infty (2k)\alpha_k (t-2)^{\underline{2k-2}} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^\infty (2k+1)\beta_k (t-2)^{\underline{2k-1}},
\end{aligned}$$

Reindexing each sum to start at $k=1$, we obtain

$$\begin{aligned}
\mathbf{J}_\nu(t-2) &= \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^\infty \alpha_{k-1} (t-2)^{\underline{2k-2}} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^\infty \beta_{k-1} (t-2)^{\underline{2k-1}}, \\
\Delta\mathbf{J}_\nu(t-1) &= \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^\infty (2k)\alpha_k (t-1)^{\underline{2k-1}} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^\infty (2k+1)\beta_k (t-1)^{\underline{2k}} + \sin\left(\frac{\pi\nu}{2}\right)\beta_0, \\
\mathbf{J}_\nu(t) &= \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^\infty \alpha_k t^{\underline{2k}} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^\infty \beta_k t^{\underline{2k+1}} + \cos\left(\frac{\pi\nu}{2}\right)\alpha_0 + t\sin\left(\frac{\pi\nu}{2}\right)\beta_0
\end{aligned}$$

Putting these into the difference equation,

$$
\begin{aligned}
& t^{\underline{2}}(\cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^{\infty}(2k)(2k-1)\alpha_k(t-2)^{\underline{2k-2}} \\
& + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}(2k+1)(2k)\beta_k(t-2)^{\underline{2k-1}}) \\
& + t(\cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^{\infty}(2k)\alpha_k(t-1)^{\underline{2k-1}} \\
& + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^{\infty}(2k+1)\beta_k(t-1)^{\underline{2k}} + \sin\left(\frac{\pi\nu}{2}\right)\beta_0) \\
& + t^{\underline{2}}(\cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^{\infty}\alpha_{k-1}(t-2)^{\underline{2k-2}} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\beta_{k-1}(t-2)^{\underline{2k-1}}) \\
& - \nu^2(\cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^{\infty}\alpha_k t^{\underline{2k}} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^{\infty}\beta_k t^{\underline{2k+1}}) \\
& + \cos\left(\frac{\pi\nu}{2}\right)\alpha_0 + \sin\left(\frac{\pi\nu}{2}\right)t\beta_0 = \frac{(t-\nu)\sin(\pi\nu)}{\pi}
\end{aligned}
$$

The equation can then be simplified to

$$
\begin{aligned}
& \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^{\infty}[(2k)(2k-1)\alpha_k + (2k)\alpha_k + \alpha_{k-1} - \nu^2\alpha_k]t^{\underline{2k}} \\
& + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}[(2k+1)(2k)\beta_k + (2k+1)\beta_k + \beta_{k-1} - \nu^2\beta_k](t-2)^{\underline{2k-1}}) \\
& + t\sin\left(\frac{\pi\nu}{2}\right)\beta_0 - \nu^2(\cos\left(\frac{\pi\nu}{2}\right)\alpha_0 + \sin\left(\frac{\pi\nu}{2}\right)t\beta_0) = \frac{(t-\nu)\sin(\pi\nu)}{\pi}
\end{aligned}
$$

It is useful to realize that

$$
\begin{aligned}
\alpha_{k-1} &= \frac{(-1)^{k-1}}{4^{k-1}\Gamma(k-1+\frac{\nu}{2}+1)\Gamma(k-1-\frac{\nu}{2}+1)} \\
&= \frac{-(-1)^k(4)(k+\frac{\nu}{2})(k-\frac{\nu}{2})}{4^k\Gamma(k+\frac{\nu}{2}+1)\Gamma(k-\frac{\nu}{2}+1)} \\
&= -4(k+\frac{\nu}{2})(k-\frac{\nu}{2})\frac{(-1)^k}{4^k\Gamma(k+\frac{\nu}{2}+1)\Gamma(k-\frac{\nu}{2}+1)} \\
&= -4(k+\frac{\nu}{2})(k-\frac{\nu}{2})\alpha_k
\end{aligned}
$$

and similarly

$$
\beta_{k-1} = -4(k+\frac{\nu}{2}+\frac{1}{2})(k-\frac{\nu}{2}+\frac{1}{2})\beta_k
$$

Substituting this in, we get

$$\cos\left(\frac{\pi\nu}{2}\right)\sum_{k=1}^{\infty}[(2k)(2k-1)\alpha_k + (2k)\alpha_k + -4(k+\frac{\nu}{2})(k-\frac{\nu}{2})\alpha_k - \nu^2\alpha_k]t^{\underline{2k}}$$
$$+\sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}[(2k+1)(2k)\beta_k + (2k+1)\beta_k$$
$$+ -4(k+\frac{\nu}{2}+\frac{1}{2})(k-\frac{\nu}{2}+\frac{1}{2})\beta_k - \nu^2\beta_k](t-2)^{\underline{2k-1}})$$
$$+ t\sin\left(\frac{\pi\nu}{2}\right)\beta_0 - \nu^2(\cos\left(\frac{\pi\nu}{2}\right)\alpha_0 + \sin\left(\frac{\pi\nu}{2}\right)t\beta_0) = \frac{(t-\nu)\sin(\pi\nu)}{\pi}$$

The summation terms simplify to zero. Using the idea that

$$\begin{aligned}\beta_0 &= \frac{1}{2\Gamma(\frac{\nu}{2}+\frac{3}{2})\Gamma(-\frac{\nu}{2}+\frac{3}{2})}\\ &= \frac{1}{2(\frac{\nu}{2}+\frac{1}{2})(\frac{\nu}{2}-\frac{1}{2})\Gamma(\frac{\nu}{2}+\frac{1}{2})\Gamma(-\frac{\nu}{2}+\frac{3}{2})}\\ &= \frac{2\sin\left(\frac{\pi\nu}{2}-\frac{\pi}{2}\right)}{(\nu^2-1)\pi}\\ &= \frac{2\cos\left(\frac{\pi\nu}{2}\right)}{(1-\nu^2)\pi}\end{aligned}$$

and similarly

$$\alpha_0 = \frac{2\sin\left(\frac{\pi\nu}{2}\right)}{\pi}$$

The equation becomes

$$\frac{z(2\sin\left(\frac{\pi\nu}{2}\right)\cos\left(\frac{\pi\nu}{2}\right))}{\pi} - \frac{\nu(2\sin\left(\frac{\pi\nu}{2}\right)\cos\left(\frac{\pi\nu}{2}\right)}{\pi} = \frac{(z-\nu)\sin(\pi\nu))}{\pi}, \qquad (13)\ \{?\}$$

which is true by the double angle formula for sine.

□

Theorem 2. *The continuous Anger function recurrence relation given in Equation 8 yields a discrete counterpart.*

$$t\boldsymbol{J}_{\nu-1}(t-1) + t\boldsymbol{J}_{\nu+1}(t-1) = 2\nu\boldsymbol{J}_{\nu}(t) - \frac{2\sin(\pi\nu)}{\pi} \qquad (14)\ \{?\}$$

Proof. Note that

$$\mathbf{J}_{\nu-1}(t) = \cos\left(\frac{\pi(\nu-1)}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{\underline{2k}}}{4^k\Gamma(k+\frac{\nu-1}{2}+1)\Gamma(k-\frac{\nu-1}{2}+1)}$$
$$+\sin\left(\frac{\pi(\nu-1)}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{\underline{2k+1}}}{2^{k+1}\Gamma(k+\frac{\nu-1}{2}+\frac{3}{2})\Gamma(k-\frac{\nu-1}{2}+\frac{3}{2})}$$

and

$$\mathbf{J}_{\nu+1}(t) = \cos\left(\frac{\pi(\nu+1)}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{4^k\Gamma(k+\frac{\nu+1}{2}+1)\Gamma(k-\frac{\nu+1}{2}+1)} + \sin\left(\frac{\pi(\nu+1)}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{2^{k+1}\Gamma(k+\frac{\nu+1}{2}+\frac{3}{2})\Gamma(k-\frac{\nu+1}{2}+\frac{3}{2})}.$$

We solve the right hand side of the recurrence relations equation.

$$\begin{aligned} &t\mathbf{J}_{\nu-1}(t-1) + t\mathbf{J}_{\nu+1}(t-1) \\ &= \cos\left(\frac{\pi(\nu-1)}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{4^k\Gamma(k+\frac{\nu-1}{2}+1)\Gamma(k-\frac{\nu-1}{2}+1)} \\ &+ \sin\left(\frac{\pi(\nu-1)}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{2^{k+1}\Gamma(k+\frac{\nu-1}{2}+\frac{3}{2})\Gamma(k-\frac{\nu-1}{2}+\frac{3}{2})} \\ &+ \cos\left(\frac{\pi(\nu+1)}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{4^k\Gamma(k+\frac{\nu+1}{2}+1)\Gamma(k-\frac{\nu+1}{2}+1)} \\ &+ \sin\left(\frac{\pi(\nu+1)}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{2^{k+1}\Gamma(k+\frac{\nu+1}{2}+\frac{3}{2})\Gamma(k-\frac{\nu+1}{2}+\frac{3}{2})}. \end{aligned}$$

Using trigonometric identities, we get

$$\begin{aligned} &\sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{4^k\Gamma(k+\frac{\nu-1}{2}+1)\Gamma(k-\frac{\nu-1}{2}+1)} \\ &- \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{2^{k+1}\Gamma(k+\frac{\nu-1}{2}+\frac{3}{2})\Gamma(k-\frac{\nu-1}{2}+\frac{3}{2})} \\ &- \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{4^k\Gamma(k+\frac{\nu+1}{2}+1)\Gamma(k-\frac{\nu+1}{2}+1)} \\ &+ \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{2^{k+1}\Gamma(k+\frac{\nu+1}{2}+\frac{3}{2})\Gamma(k-\frac{\nu+1}{2}+\frac{3}{2})}. \end{aligned}$$

Using Gamma identities and algebra yields

$$2\nu\cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{4^k\Gamma(k+\frac{\nu}{2}+1)\Gamma(k-\frac{\nu}{2}+1)} + 2\nu\sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{2^{2k+1}\Gamma(k+\frac{\nu}{2}+\frac{3}{2})\Gamma(k-\frac{\nu}{2}+\frac{3}{2})} - \frac{2\sin(\pi\nu)}{\pi}.$$

□

Corollary 2.1. *The Anger function recurrence relation given in Equation 9 yields a discrete counterpart*

$$\boldsymbol{J}_{\nu-1}(t) + \boldsymbol{J}_{\nu+1}(t) = 2\Delta \boldsymbol{J}_{\nu}(t). \tag{15}$$ {?}

Corollary 2.2. *Similarly, the Anger function recurrence relation given in Equation 10 yields a discrete counterpart*

$$t\Delta \boldsymbol{J}_{\nu}(t-1) \pm \nu \boldsymbol{J}_{\nu}(t) = \pm t \boldsymbol{J}_{\nu\mp 1}(t-1) \pm \frac{\sin(\pi\nu)}{\pi}. \tag{16}$$ {?}

The Laplace transform of continuous Anger function is

$$\begin{aligned}\mathscr{L}(\mathfrak{J}_{\nu}(z)) = {} & \frac{\cos\frac{\pi\nu}{2}}{s\Gamma(1-\frac{\nu}{2})\Gamma(1+\frac{\nu}{2})}\,{}_3F_2\left(1,1,\frac{1}{2};1-\frac{1}{2}\nu,1+\frac{1}{2}\nu;-\frac{1}{s^2}\right) \\ & + \frac{\sin\frac{\pi\nu}{2}}{2s^2\Gamma(\frac{3-\nu}{2})\Gamma(\frac{3+\nu}{2})}\,{}_3F_2\left(1,1,\frac{3}{2};\frac{1}{2}(3-\nu),\frac{1}{2}(3+\nu);-\frac{1}{s^2}\right). \end{aligned} \tag{17}$$ {?}

We now derive the Laplace transform of the discrete Anger function.

Theorem 3.

$$\begin{aligned}\mathscr{L}\{\boldsymbol{J}_{\nu}(t)\} = {} & \frac{\cos\frac{\pi\nu}{2}}{s\Gamma(1-\frac{\nu}{2})\Gamma(1+\frac{\nu}{2})}\,{}_3F_2\left(1,1,\frac{1}{2};1-\frac{1}{2}\nu,1+\frac{1}{2}\nu;-\frac{1}{s^2}\right) \\ & + \frac{\sin\frac{\pi\nu}{2}}{2s^2\Gamma(\frac{3-\nu}{2})\Gamma(\frac{3+\nu}{2})}\,{}_3F_2\left(1,1,\frac{3}{2};\frac{1}{2}(3-\nu),\frac{1}{2}(3+\nu);-\frac{1}{s^2}\right). \end{aligned} \tag{18}$$ {?}

Proof. Let

$$\lambda = \frac{\cos\frac{\pi\nu}{2}}{\Gamma(1-\frac{\nu}{2})\Gamma(1+\frac{\nu}{2})}$$
$$\omega = \frac{\sin\frac{\pi\nu}{2}}{2\Gamma(\frac{3-\nu}{2})\Gamma(\frac{3+\nu}{2})}$$

Then,

$$\mathscr{L}\{\mathbf{J}_\nu(t)\} = \lambda \sum_{k=0}^{\infty} \frac{(1)_k(\frac{-1}{4})^k}{(1-\frac{n}{2})_k(1+\frac{n}{2})_k k!} \mathscr{L}\{\underline{t^{2k}}\}$$

$$+ \omega \sum_{k=0}^{\infty} \frac{(1)_k(\frac{-1}{4})^k}{(\frac{3-n}{2})_k(\frac{3+n}{2})_k k!} \mathscr{L}\{\underline{t^{2k+1}}\}$$

$$= \lambda \sum_{k=0}^{\infty} \frac{(1)_k(\frac{-1}{4})^k}{(1-\frac{n}{2})_k(1+\frac{n}{2})_k k!} \frac{(\frac{1}{2})_k(2)^k(2)^k(1)_k}{s^{2k+1}}$$

$$+ \omega \sum_{k=0}^{\infty} \frac{(1)_k(\frac{-1}{4})^k}{(\frac{3-n}{2})_k(\frac{3+n}{2})_k k!} \frac{(\frac{3}{2})_k(2)^k(2)^k(1)_k}{s^{2k+2}}$$

$$= \frac{1}{s}\lambda \sum_{k=0}^{\infty} \frac{(1)_k(1)_k(\frac{-1}{s^2})^k(\frac{1}{2})_k}{(1-\frac{n}{2})_k(1+\frac{n}{2})_k k!} + \frac{1}{2s^2}\omega \frac{(1)_k(1)_k(\frac{3}{2})_k(\frac{-1}{s^2})^k}{(\frac{3-n}{2})_k(\frac{3+n}{2})_k k!}$$

$$= \frac{\cos\frac{\pi\nu}{2}}{s\Gamma(1-\frac{\nu}{2})\Gamma(1+\frac{\nu}{2})} {}_3F_2\left(1,1,\frac{1}{2};1-\frac{1}{2}\nu,1+\frac{1}{2}\nu;-\frac{1}{s^2}\right)$$

$$+ \frac{\sin\frac{\pi\nu}{2}}{2s^2\Gamma(\frac{3-\nu}{2})\Gamma(\frac{3+\nu}{2})} {}_3F_2\left(1,1,\frac{3}{2};\frac{1}{2}(3-\nu),\frac{1}{2}(3+\nu);-\frac{1}{s^2}\right).$$

□

4 Conclusion

strange situation with noninteger ν for J_ν – in that case the J_ν function is identically zero and so the "usual" tricks for relating oscillation of a homogeneous difference equation to a nonhomogeneous "forced" difference equation are not applicable (see bohner oscillation book section 1.15)

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