# Bate's Triangle

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## 1 Introduction

Over the course of this research project, our team has been examining the intricacies of the mathematical object that has been lovingly and not-at-all-narcissistically dubbed "Bate's triangle." This mathematical object takes the form of a triangular array of natural numbers, much like Pascal's triangle, and it can in fact be generated in a very similar manner. Unlike its more docile cousin, however, Bate's triangle contains a number of perplexities that have yet obstinately eluded full understanding - most notably, a set of patterns found in the prime factorization of the elements that appear to have fractal properties similar to the Sierpinski triangle. In addition, Bate's triangle also has connections to Chebyshev polynomials and the Riemann Zeta function (the latter of which is clearly so self-evident that it needn't be mentioned at all in this paper).

The goal of our investigation is to identify and describe some of the patterns that appear in Bate's triangle, and hopefully come to some understanding of why these patterns appear.

## 2 Definitions

### 2.1 Bate's Triangle: The Simple Formulation

Bate's Triangle can be formed in a manner very similar to that of Pascal's Triangle. In Pascal's Triangle, we place *1*'s along the sides of the triangle, and each internal entry is formed by adding the two entries above it, as seen below.

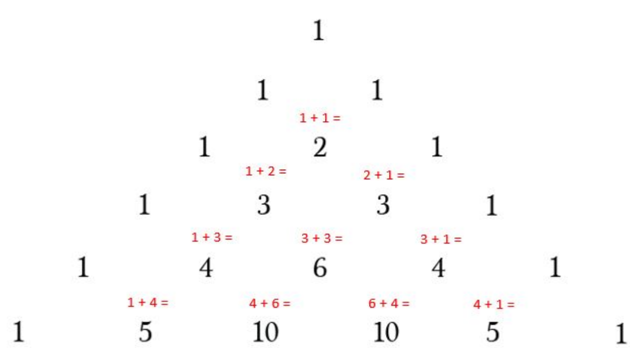



Figure 1: Pascal's Triangle[1]

The difference in Bate's triangle is that the entries down the sides of the triangle are different. Down the left side, we have alternating *0*'s and *1*'s, starting with *0* at the top. On the right side, we simply add *1* to the previous entry on that side; we are, in essence, just counting upward as we go down the right side of the triangle. The internal entries are just like Pascal's triangle – we add the two entries above to get the "current entry."

### 2.2 Bate's Triangle: The Not-so-simple Formulation

Calculating elements of Bate's triangle as the sum of previous entries is not the only way to generate the numbers, and this method may in fact obscure some of the connections between the triangle and other areas

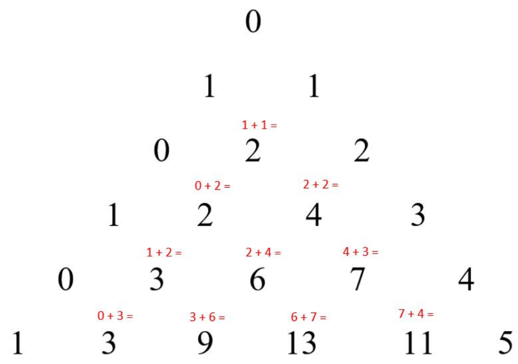



Figure 2: Bate's Triangle

of math. The elements of Bate's triangle can also be defined based on the summed coefficients of Chebyshev polynomials of the second kind.

A Chebyshev polynomial of the second kind (denoted $U_n(x)$, where $n$ is the degree) is a special type of polynomial defined using the following formula:

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}.$$

The first few Chebyshev polynomials are listed below:

$U_0(x) = 1$
$U_1(x) = 0 + 2x$
$U_2(x) = -1 + 0 + 4x^2$
$U_3(x) = 0 - 4x + 0 + 8x^3$
$\dots$

Now suppose a certain individual wanted to take the sum of the first $N$ Chebyshev polynomials (why anyone would think to do this is beyond the scope of this paper):

$$\sum_{n=0}^{N} U_n(x) = \sum_{n=0}^{N} \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k} \right).$$

In order to derive an expression for just the summed coefficients, an application of change of summation is used to move the $x$ term to the outer summation. This yields:

$$\sum_{n=0}^{N} U_n(x) = \sum_{j=0}^{N} \left( \sum_{k=0}^{\lfloor \frac{N-j}{2} \rfloor} (-1)^k \binom{j+k}{k} \right) (2x)^j$$

where $j = n - 2k$. We can take these summed coefficients and arrange them into a triangle (see Figure 3).

In this triangle, item $j$ or row $m$ can be expressed as:

$$\begin{bmatrix} m \\ j \end{bmatrix} = \sum_{k=0}^{\lfloor \frac{m-j}{2} \rfloor} (-1)^k \binom{j+k}{k}.$$

Note from Figure 3 that this triangle has duplicate diagonals. We have taken the liberty of removing them in order to better see the patterns present. This is done by removing the floor function on the upper

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | | | | | | | | |
| 0 | 1 | 1 | | | | | | | | | | | | | | | | |
| 0 | -1 | 1 | 1 | | | | | | | | | | | | | | | |
| 1 | -1 | -2 | 1 | 1 | | | | | | | | | | | | | | |
| 1 | 2 | -2 | -3 | 1 | 1 | | | | | | | | | | | | | |
| 0 | 2 | 4 | -3 | -4 | 1 | 1 | | | | | | | | | | | | |
| 0 | -2 | 4 | 7 | -4 | -5 | 1 | 1 | | | | | | | | | | | |
| 1 | -2 | -6 | 7 | 11 | -5 | -6 | 1 | 1 | | | | | | | | | | |
| 1 | 3 | -6 | -13 | 11 | 16 | -6 | -7 | 1 | 1 | | | | | | | | | |
| 0 | 3 | 9 | -13 | -24 | 16 | 22 | -7 | -8 | 1 | 1 | | | | | | | | |
| 0 | -3 | 9 | 22 | -24 | -40 | 22 | 29 | -8 | -9 | 1 | 1 | | | | | | | |
| 1 | -3 | -12 | 22 | 46 | -40 | -62 | 29 | 37 | -9 | -10 | 1 | 1 | | | | | | |
| 1 | 4 | -12 | -34 | 46 | 86 | -62 | -91 | 37 | 46 | -10 | -11 | 1 | 1 | | | | | |
| 0 | 4 | 16 | -34 | -80 | 86 | 148 | -91 | -128 | 46 | 56 | -11 | -12 | 1 | 1 | | | | |
| 0 | -4 | 16 | 50 | -80 | -166 | 148 | 239 | -128 | -174 | 56 | 67 | -12 | -13 | 1 | 1 | | | |
| 1 | -4 | -20 | 50 | 130 | -166 | -314 | 239 | 367 | -174 | -230 | 67 | 79 | -13 | -14 | 1 | 1 | | |
| 1 | 5 | -20 | -70 | 130 | 296 | -314 | -553 | 367 | 541 | -230 | -297 | 79 | 92 | -14 | -15 | 1 | 1 | |
| 0 | 5 | 25 | -70 | -200 | 296 | 610 | -553 | -920 | 541 | 771 | -297 | -376 | 92 | 106 | -15 | -16 | 1 | 1 |
| 0 | -5 | 25 | 95 | -200 | -496 | 610 | 1163 | -920 | -1461 | 771 | 1068 | -376 | -468 | 106 | 121 | -16 | -17 | 1 |
| 1 | -5 | -30 | 95 | 295 | -496 | -1106 | 1163 | 2083 | -1461 | -2232 | 1068 | 1444 | -468 | -574 | 121 | 137 | -17 | -18 |

Figure 3: Summed coefficients

bound of the summation, yielding:

$$\begin{bmatrix} m \\ j \end{bmatrix} = \sum_{k=0}^{m-j} (-1)^k \binom{j+k}{k}.$$

As a further simplification, we can introduce the term $(-1)^{j+m}$ to make all the values positive (though in the investigation of certain patterns we have chosen to keep the negatives). This gives us the final formula that defines any element in Bate's triangle:

$$\begin{bmatrix} m \\ j \end{bmatrix} = (-1)^{j+m} \sum_{k=0}^{m-j} (-1)^k \binom{j+k}{k}.$$

# 3 The Problem

As this triangle is new, with no information about it readily available, we were mainly concerned with determining what features the triangle had. Our research on this was focused on finding patterns and relating Bate's Triangle to Pascal's Triangle, as both are generated in a very similar way.

Partway into the semester, we decided to reverse the generation of Bate's Triangle, figuring out what the numbers above it would be. This resulted in a string of alternating 1s and $-1$s along the left side, and a string of 1s along the right side. We eventually decided that the right-side border of 1s should be a part of the triangle, although most of our research did not include it.

In looking at these borders, we discovered that we could form a hexagon from four copies of Bate's Triangle (two of which were mirror-images) and two copies of Pascal's Triangle. All six triangles share the same borders of 1s and alternating 1s. This led to a question about how Bate's Triangle relates to Pascal's Triangle. In our hexagon, each copy of Pascal's Triangle, which is symmetric, was flanked by a copy of Bate's Triangle on each side. However, we were unable to see how the two triangles related in this manner.

# 4 The Results

## 4.1 Row Sums

One well-known feature of Pascal's Triangle is that the rows sum up to $2^n$ where $n$ is the row number. When we checked the row sums for Bate's Triangle, we discovered that there are two types of rows. One type of row nicely doubles the sum of the last, as in Pascal's Triangle, but the other row type doubles the sum of the last and adds two.

The row types alternate because the triangle generation effectively doubles each row as in Pascal's Triangle. However, it would ordinarily be one more than double, but since the left side alternates between *0* and

*1*, the even-numbered rows lose this extra number above double. The odd-numbered rows gain two more than double, as they gain one when switching from *0* to *1* on the left side, in addition to gaining another one from the "hidden" *1* border on the right side.

In creating a formula for this, we came up with two different types, both dependent on the row number. In both cases, $n$ is the number of the row. For the row-doubling type, which is even-numbered rows:

$$2(1+\frac{4^n-4}{3})$$

And for the other type of row, which is for odd-numbered rows:

$$\frac{4^n-4}{3}$$

## 4.2 Hockey Stick

Not surprisingly, due to the non-symmetrical nature of Bate's Triangle, most of the patterns that can be found in Pascal's Triangle do not carry over. One of the ones that we attempted was the hexagon pattern. In Pascal's Triangle, if any ring of six numbers is chosen, anywhere in the triangle, the sums of the alternate numbers are equal to each other. This was not true in Bate's Triangle, and there was no clear way to determine the difference, either.

The one pattern that did seem to hold in Bate's Triangle is the"hockey stick pattern." This pattern is generated by summing the first $n$ terms of any diagonal (beginning from the outside 1). This sum is equal to the next term in the diagonal that intersects the last term on the first diagonal, forming a"hockey stick" shape. In Bate's Triangle, the term on the second diagonal does not equal the sum of the first diagonal terms. Instead, the second diagonal term is either one more, when starting with a *1* from the left side, or one less, when starting from the right side or from a *0* on the left side.

## 4.3 Fractals

### 4.3.1 Prime Factorization

One interesting aspect of the triangle we explored was the prime factorization of each entry. We generated these numbers via a Python script[2], and highlighted each appearance of a prime number with different colors to look for patterns.

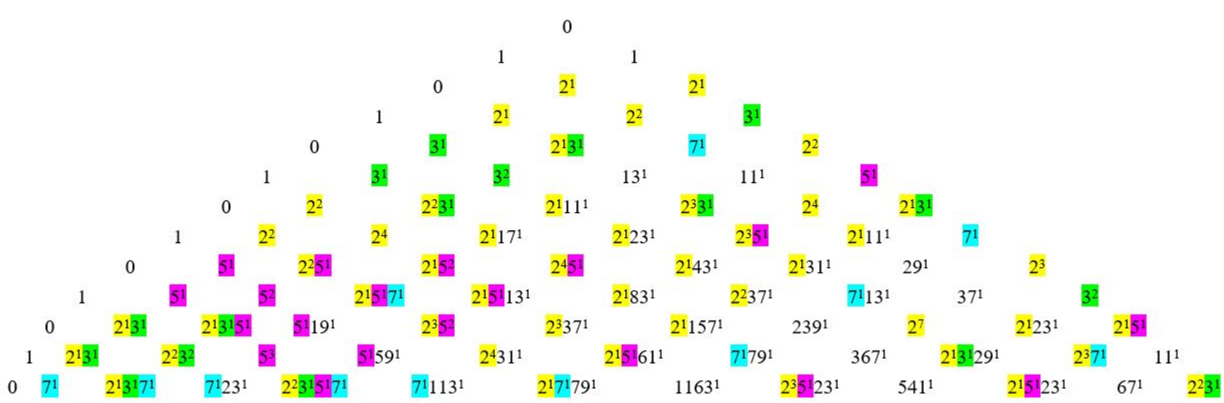



Figure 4: Prime Factorization

We noticed a pattern among those entries that contained a *2*. Since the prime factorization is based on division, and *2* divides *0*, we highlighted all the *0*'s along with the *2*'s and got the following:

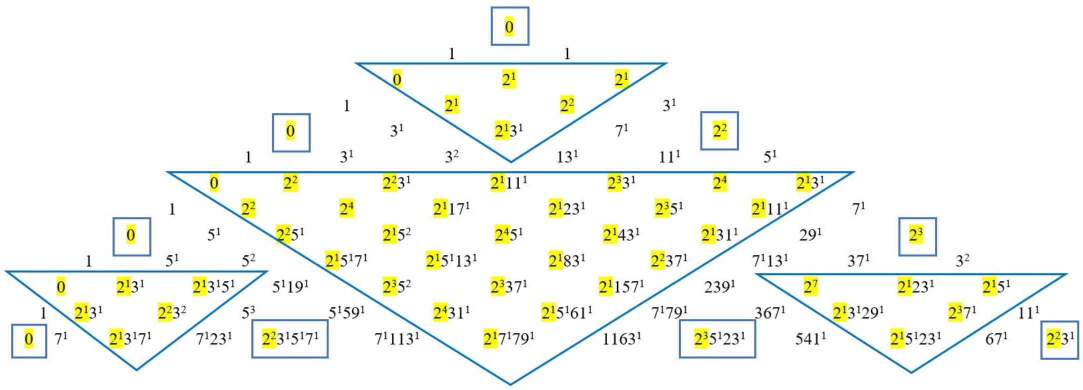



Figure 5: Prime Factorization

#### 4.3.2 Fractals

This looked like it might be a fractal. We used Mathematica to generate another image to explore the pattern more easily. Essentially, we divided every entry of the triangle by *2*. If the number was divisible by *2*, we made that entry black; otherwise, it was yellow. (Note: for the figure below, all the entries down the right side of the triangle are yellow, indicating that they are not divisible by *2*. The reason for this is that we added *1*'s down the right side of the triangle as a sort of "padding." This padding, however, does not change our general rule for generating internal entries of the triangle.)

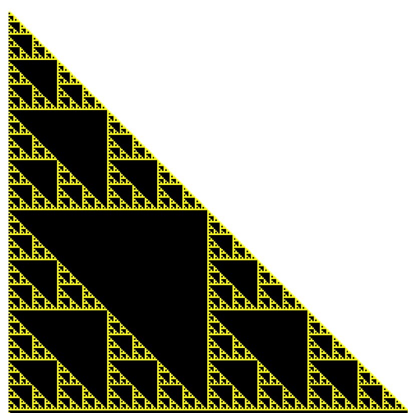


Figure 6: "*2*'s" Fractal

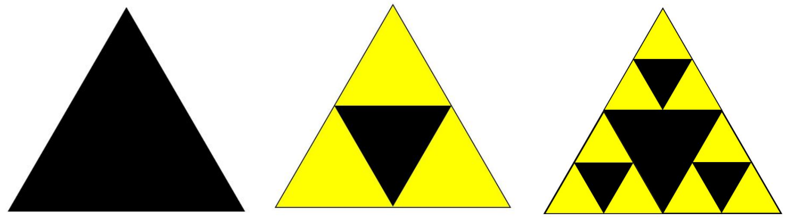
We were able to find a rule for how to generate this fractal. We converted the triangle to an equilateral triangle to make the pattern more clear. Our first "base pattern" is a simple black triangle. For the second base pattern, which is the primary base pattern, we begin with a triangle split evenly into four subtriangles with the center triangle black. This is the basic pattern of the triangle. To create the fractal, we repeat this same primary base pattern in all the yellow triangles. We can repeat this pattern as much as we like.

We generated several other fractal-like images with the same technique mentioned above (diving every entry in the triangle by a number $n$ and coloring according to whether or not the entry was divisible by $n$). The following are the other images generated.

In our work so far, we were able to figure out the rules for generating the blue triangle ($n = 3$) and made notes about the rules for generating the red triangle ($n = 5$).

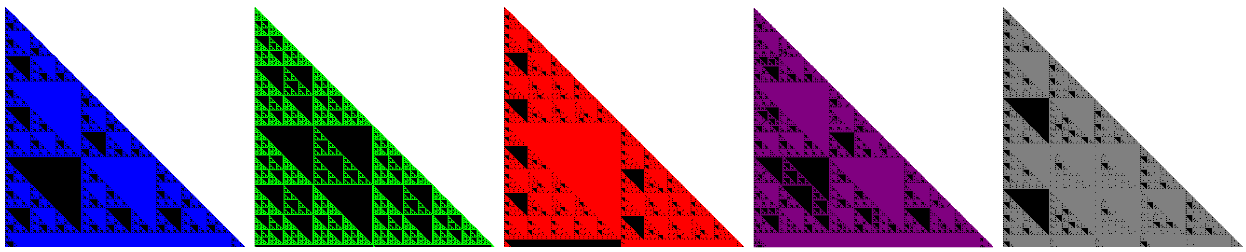
For the blue triangle (triangle (a) from Figure 7), which we will call 3's Fractal, we identified three base triangles that can be used to generate the fractal.

To describe how these base triangles appear in the fractal, we define three rotations to be used on base triangle (c) from Figure 8:



(a) Base Pattern (b) Primary Base Pattern (c) Base Pattern Applied

Figure 7: 2's Pattern



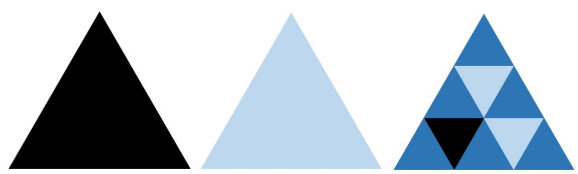
(a) Divide by 3 (b) Divide by 4 (c) Divide by 5 (d) Divide by 6 (e) Divide by 7

Figure 8: Other Fractal-like Patterns

$e$ - the identity function

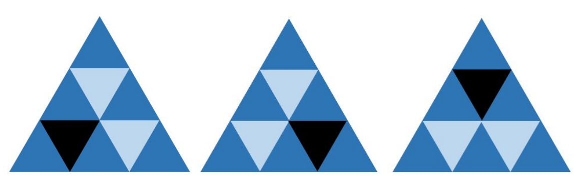
$g$ - we rotate the triangle 120° counterclockwise

$g^2$ - we rotate the triangle 240° counterclockwise



(a) (b) (c)

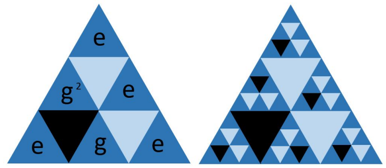
Figure 9: Base Patterns for 3's Fractal



(a) $e$ (b) $g$ (c) $g^2$

Figure 10: Functions applied to base pattern (c)

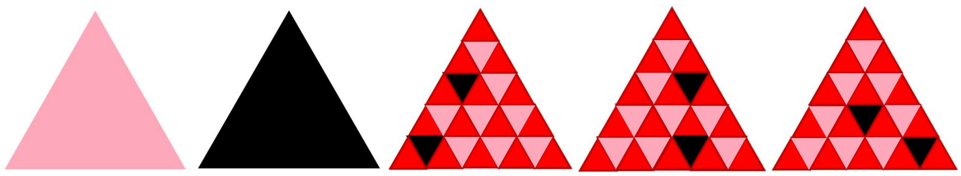
To generate the fractal, we start with triangle (c) and leave the light blue and black triangles as they are. We then apply functions to a smaller triangle (c) and place them in the big triangle following this guide.




(a) (b)

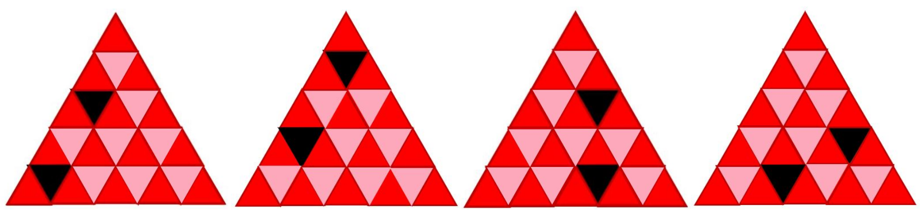
Figure 11: Forming the 3's Triangle

We were able to find a similar set of base triangles for the 5's Fractal (the red triangle in Figure 7). There were five of them as seen below. We use triangle (c) as our main triangle. We again define a set



(a) (b) (c) (d) (e)

Figure 12: Base Patterns for 5's Fractals



(a) Base Triangle (c) (b) $gh$(c) (c) Base Triangle (d) (d) $gh$(d)

Figure 13: Transformations applied to 5's Fractal base triangles

of transformations to perform on the base triangles of the triangle. This is a bit more complex than the previous example because we are now performing the transformations on multiple base patterns instead of just one. We can see, generally, how this fractal plays out when we apply these transformations to our base triangles.

$e$ - the identity function

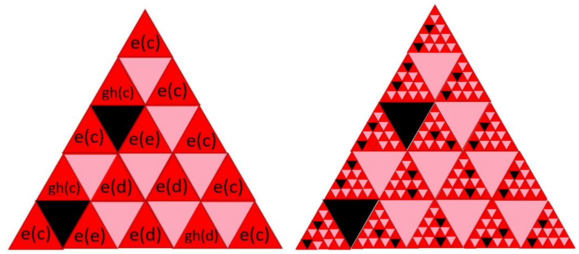
$h$ - we do a horizontal flip on the triangle

$g$ - we rotate the triangle 120° counterclockwise

We use triangle (c) as our main base triangle and apply various functions to various base triangles throughout the triangle to get the first form of our triangle.

As we continued to explore this fractal, we saw that there are a series of transformations that must be applied to the subtriangles of base triangles (d) and (e). These observations are beyond the scope of this paper.

As we progressed through our study of the fractals, we made some observations that applied to the three on which we focused. If we let $n$ be the number by which we divided each element of the original triangle (padded with *1*'s) to generate the fractal, then, based on our observations, a fractal that was formed by dividing by $n$ has




(a) (b)

Figure 14: The functions on the 5 Fractal applied

1. $n$ base patterns, and
2. $n^2$ subtriangles in its base pattern.

# 5 Possible Extensions

Although we have only observed these properties in the first three fractals, we suspect that this pattern will continue with other $n$'s. One possible extension of this work would be to continue to discover the base patterns and transformations that generate these fractals. Another extension would be write formal proofs for all our observations.

# 6 References

1. Pascal's Triangle original image. https://medium.com/i-math/top-10-secrets-of-pascals-triangle-6012ba9c5e23
2. The script to generate the triangle and its prime factorization. https://github.com/rachaelrogan/Bates-Triangle.