

The Polynomial Shift Operator on the Quantum Time Scale

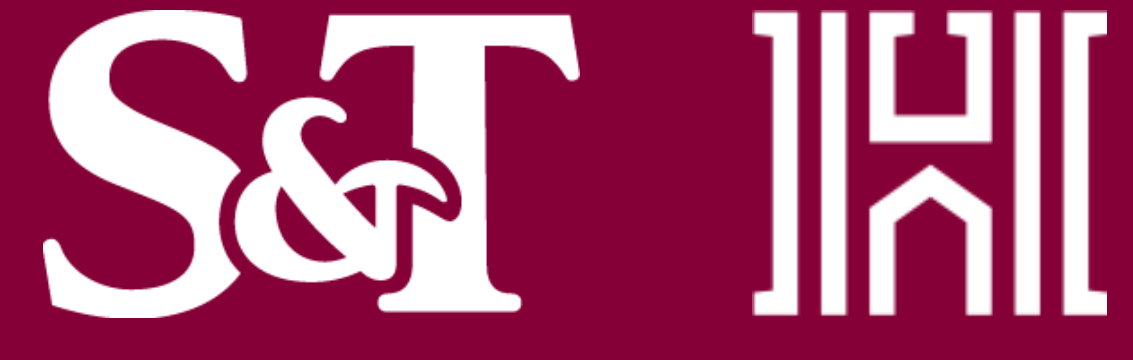
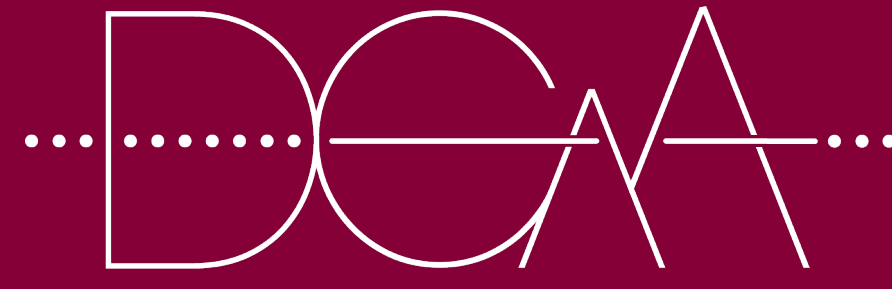
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Introduction

The goal of our study is to investigate the **polynomial shift operator** $\mathcal{U}$ on the **quantum time scale** $q^{\mathbb{N}_0}$. This allows us to generalize results from calculus to more restricted domains.

Unifying Analysis

Time scale calculus is a unification of **continuous analysis** (calculus on the real line) and **discrete analysis** (calculus on the integers).

TYPE OF ANALYSIS	DOMAIN	FUNDAMENTAL OPERATION
Continuous Analysis	$\mathbb{R}$ Real line	$f'(t)$ Derivative
Discrete Analysis	$\mathbb{Z}$ Integers	$\Delta f(t)$ Difference
Time Scale Calculus	$\mathbb{T}$ Time scale	$f^\Delta(t)$ Delta-derivative

A time scale is **any closed subset of the real numbers**.

- Common time scales:
 $\mathbb{R}$ $\mathbb{Z}$ $[0, 1]$
- Time scale operations:
 - Forward jump operator** $\sigma(t)$: next element of the time scale
 - Ex: $\sigma(t) = t + 1$ in $\mathbb{Z}$
 - Graininess** $\mu(t) = \sigma(t) - t$: distance to next element of the time scale
 - Ex: $\mu(t) = 1$ in $\mathbb{Z}$
- Delta-derivative:**

$$f^\Delta(t) = \begin{cases} f'(t), & \mu(t) = 0 \\ \frac{f(\sigma(t)) - f(t)}{\mu(t)}, & \mu(t) > 0 \end{cases}$$

Redefining Powers

The **power rule of differentiation**:

$$(t^k)^\Delta = kt^{k-1}$$

In order to respect the power rule, we must redefine polynomial powers in different time scales:

$$\begin{aligned} (t, s)_{\mathbb{R}}^k &= (t - s)^k \\ &\downarrow \\ (t, s)_{\mathbb{Z}}^k &= (t - s)(t - s - 1) \dots (t - s - (t - 1)) \\ &\downarrow \\ (t, s)_{q^{\mathbb{N}_0}}^k &= \prod_{\nu=0}^{k-1} \frac{t - sq^\nu}{\sum_{\mu=0}^{\nu} q^\mu} \end{aligned}$$

$\mathcal{U}$ -Shift Operator

The polynomial shift operator $\mathcal{U}$ increases the order of a polynomial function by 1:

- On the real numbers, we multiply by a factor of $(t - s)$
 - Ex: $\mathcal{U} \{(t, s)_{\mathbb{Z}}^2\} = (t, s)_{\mathbb{Z}}(t - 1, s)_{\mathbb{Z}}^2 = (t, s)_{\mathbb{Z}}^3$
- Shift on other time scales is more complex due to redefined powers
 - Ex: $\mathcal{U} \{(t, s)_{\mathbb{R}}^2\} = \mathcal{U} \{(t - s)^2\} = (t - s)(t - s)^2 = (t - s)^3$
- General case, see Figure 1:

$$\mathcal{U}\{f\}(t) = \left(\mathcal{L}^{-1} \circ \left[-\frac{d}{dz} \right] \circ \mathcal{L} \right) \{f\}(t)$$

From the general case, we derived a functional representation for $\mathcal{U}\{f\}$ on $q^{\mathbb{N}_0}$ for arbitrary f :

$$\mathcal{U}_{q^{\mathbb{N}_0}}\{f\}(q^m) = \sum_{\aleph=1}^{m-1} \mu(q^\aleph) f(q^\aleph) A_\aleph B_{\aleph, \aleph} + \sum_{\beth=m}^{\infty} \mu(q^\beth) f(q^\beth) A_\beth [B_{m-1, \beth} + C_\beth]$$

where

$$\begin{aligned} A_x &= \left[\prod_{\alpha=0}^x \frac{1}{(q-1)q^\alpha} \right] \left[\prod_{\beta=0}^{m-1} (q-1)q^\beta \right] \\ B_{x,y} &= \sum_{\lambda=0}^x \left[\prod_{\gamma=0}^{m-1} z_\lambda - z_\gamma \right] \left[\prod_{\eta=0}^y \frac{1}{z_\lambda - z_\eta} \right] \\ C_x &= \sum_{\psi=m}^x \left[\prod_{\gamma=0}^{m-1} z_\psi - z_\gamma \right] \left[\prod_{\eta=0, (\psi)}^x \frac{1}{z_\psi - z_\eta} \right] \left[\sum_{\eta=m, (\psi)}^x \frac{-1}{z_\psi - z_\eta} \right] \end{aligned}$$

Quantum Calculus

Calculus on the **quantum time scale**

$$q^{\mathbb{N}_0} = \{1, q, q^2, q^3, \dots\}$$

where $q > 1$

- Forward jump: $\sigma(t) = qt$
- Graininess: $\mu(t) = (q - 1)t$

Connections exist to:

- Combinatorics
- Number theory
- Quantum computing
- Relativity

Laplace Transform

- The **Laplace transform** is a transform of a function from the time scale to the complex plane:

$$\mathcal{L}\{f\}(z, s) = \int_s^\infty f(\tau) e_{\ominus z}(\sigma(\tau), s) \Delta\tau$$

where $e_{\ominus z}(t, s)$ is the time scale analogue of $e^{-z(t-s)}$ with the same derivative properties

- The **inverse Laplace transform** $\mathcal{L}^{-1}$ undoes the Laplace transform
- In general, differentiation in the complex numbers corresponds to multiplication by t in the time scale

$$\begin{aligned} \sum_{k=0}^{\infty} a_k (t - s)_{\mathbb{T}}^k &\xrightarrow{\mathcal{L}_{\mathbb{T}}} \sum_{k=0}^{\infty} \frac{a_k k!}{z^{k+1}} \\ &\downarrow \mathcal{U}_{\mathbb{T}} \qquad \qquad \downarrow -\frac{d}{dz} \\ \sum_{k=0}^{\infty} a_k (t - s)_{\mathbb{T}}^{k+1} &\xleftarrow{\mathcal{L}_{\mathbb{T}}^{-1}} \sum_{k=0}^{\infty} \frac{a_k (k+1)!}{z^{k+2}} \end{aligned}$$

FIGURE 1: The mechanism of the shift operator on an arbitrary function and time scale

Alternate Approach

As an alternate approach, we consider a non-standard redefinition of a polynomial power $(t - s)^k$:

$$\delta_k(t, s) = \prod_{\nu=0}^{k-1} t - sq^\nu$$

- Easier shift property: $\delta_k(t, s) \cdot \delta_m(t, \sigma^k(s)) = \delta_{k+m}(t, s)$
- Nonlinear coefficient for power rule:

$$\delta_k^\Delta(t, s) = \frac{q^k - 1}{q - 1} \delta_{k-1}(t, s)$$

Future Works

- Investigate how various differential equations behave with δ_k
- Operator equations with $\mathcal{U}$ in quantum calculus

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