The Polynomial Shift Operator on the Quantum Time Scale

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Introduction

The goal of our study is to investigate the **polynomial shift operator** $\mathscr{U}$ on the **quantum time scale** $q^{\mathbb{N}_0}$. This allows us to generalize results from calculus to more restricted domains.

Unifying Analysis

Time scale calculus is a unification of **continuous analysis** (calculus on the real line) and **discrete analysis** (calculus on the integers).

Type of Analysis	Domain		Fundamental Operation	
Continuous Analysis	$\mathbb{R}$	Real line	$f'(t)$	Derivative
Discrete Analysis	$\mathbb{Z}$	Integers	$\Delta f(t)$	Difference
Time Scale Calculus	$\mathbb{T}$	Time scale	$f^{\Delta}(t)$	Delta-derivative

A time scale is **any closed subset of the real numbers.**

- Common time scales:
  $\mathbb{R}$ $\quad\mathbb{Z}\quad$ $[0,1]$
- Time scale operations:
  - **Forward jump operator** $\sigma(t)$: next element of the time scale
    - Ex: $\sigma(t)=t+1$ in $\mathbb{Z}$
  - **Graininess** $\mu(t)=\sigma(t)-t$: distance to next element of the time scale
    - Ex: $\mu(t)=1$ in $\mathbb{Z}$
- **Delta-derivative:**

$$f^{\Delta}(t)=\begin{cases} f'(t), & \mu(t)=0 \\ \frac{f(\sigma(t))-f(t)}{\mu(t)}, & \mu(t)>0 \end{cases}$$

Redefining Powers

The **power rule of differentiation:**

$$\left(t^k\right)^{\Delta}=kt^{k-1}$$

In order to respect the power rule, we must redefine polynomial powers in different time scales:

$$\begin{aligned} (t,s)_{\mathbb{R}}^{k} &= (t-s)^k \\ &\downarrow \\ (t,s)_{\mathbb{Z}}^{k} &= (t-s)(t-s-1)\dots(t-s-(t-1)) \\ &\downarrow \\ (t,s)_{q^{\mathbb{N}_0}}^{k} &= \prod_{\nu=0}^{k-1}\frac{t-sq^{\nu}}{\sum_{\mu=0}^{\nu}q^{\mu}} \end{aligned}$$

$\mathscr{U}$-Shift Operator

The polynomial shift operator $\mathscr{U}$ increases the order of a polynomial function by 1:

- On the real numbers, we multiply by a factor of $(t-s)$
  - Ex: $\mathscr{U}\left\{(t,s)_{\mathbb{Z}}^{2}\right\}=(t,s)_{\mathbb{Z}}(t-1,s)_{\mathbb{Z}}^{2}=(t,s)_{\mathbb{Z}}^{3}$
- Shift on other time scales is more complex due to redefined powers
  - Ex: $\mathscr{U}\left\{(t,s)_{\mathbb{R}}^{2}\right\}=\mathscr{U}\left\{(t-s)^2\right\}=(t-s)(t-s)^2=(t-s)^3$
- General case, see Figure 1:

$$\mathscr{U}\{f\}(t)=\left(\mathscr{L}^{-1}\circ\left[-\frac{\mathrm{d}}{\mathrm{d}z}\right]\circ\mathscr{L}\right)\{f\}(t)$$

From the general case, we derived a functional representation for $\mathscr{U}\{f\}$ on $q^{\mathbb{N}_0}$ for arbitrary f:

$$\mathscr{U}_{q^{\mathbb{N}_0}}\{f\}\left(q^m\right)=\sum_{\aleph=1}^{m-1}\mu\left(q^{\aleph}\right)f\left(q^{\aleph}\right)A_{\aleph}B_{\aleph,\aleph}+\sum_{\beth=m}^{\infty}\mu\left(q^{\beth}\right)f\left(q^{\beth}\right)A_{\beth}\left[B_{m-1,\beth}+C_{\beth}\right]$$

where

$$A_x=\left[\prod_{\alpha=0}^{x}\frac{1}{(q-1)q^{\alpha}}\right]\left[\prod_{\beta=0}^{m-1}(q-1)q^{\beta}\right]$$

$$B_{x,y}=\sum_{\lambda=0}^{x}\left[\prod_{\gamma=0}^{m-1}z_{\lambda}-z_{\gamma}\right]\left[\prod_{\eta=0}^{y}\frac{1}{z_{\lambda}-z_{\eta}}\right]$$

$$C_x=\sum_{\psi=m}^{x}\left[\prod_{\gamma=0}^{m-1}z_{\psi}-z_{\gamma}\right]\left[\prod_{\eta=0,(\psi)}^{x}\frac{1}{z_{\psi}-z_{\eta}}\right]\left[\sum_{\eta=m,(\psi)}^{x}\frac{-1}{z_{\psi}-z_{\eta}}\right]$$

$$\begin{tikzcd} \sum_{k=0}^{\infty}a_k(t-s)_{\mathbb{T}}^{k} \arrow[r, "\mathscr{L}_{\mathbb{T}}"] \arrow[d, "\mathscr{U}_{\mathbb{T}}"] & \sum_{k=0}^{\infty}\frac{a_k k!}{z^{k+1}} \arrow[d, "-\frac{\mathrm{d}}{\mathrm{d}z}"] \\ \sum_{k=0}^{\infty}a_k(t-s)_{\mathbb{T}}^{k+1} & \sum_{k=0}^{\infty}\frac{a_k(k+1)!}{z^{k+2}} \arrow[l, "\mathscr{L}_{\mathbb{T}}^{-1}"] \end{tikzcd}$$

Figure 1: The mechanism of the shift operator on an arbitrary function and time scale

Alternate Approach

As an alternate approach, we consider a non-standard redefinition of a polynomial power $(t-s)^k$:

$$\delta_k(t,s)=\prod_{\nu=0}^{k-1}t-sq^{\nu}$$

- Easier shift property: $\delta_k(t,s)\cdot\delta_m\left(t,\sigma^k(s)\right)=\delta_{k+m}(t,s)$
- Nonlinear coefficient for power rule:

$$\delta_k^{\Delta}(t,s)=\frac{q^k-1}{q-1}\delta_{k-1}(t,s)$$

Quantum Calculus

Calculus on the **quantum time scale**

$q^{\mathbb{N}_0}=\{1,q,q^2,q^3,\dots\}$

where $q>1$

- Forward jump: $\sigma(t)=qt$
- Graininess: $\mu(t)=(q-1)t$

Connections exist to:

- Combinatorics
- Number theory
- Quantum computing
- Relativity

Laplace Transform

- The **Laplace transform** is a transform of a function from the time scale to the complex plane:

$$\mathscr{L}\{f\}(z,s)=\int_{s}^{\infty}f(\tau)e_{\ominus z}(\sigma(\tau),s)\Delta\tau$$

  where $e_{\ominus z}(t,s)$ is the time scale analogue of $e^{-z(t-s)}$ with the same derivative properties
- The **inverse Laplace transform** $\mathscr{L}^{-1}$ undoes the Laplace transform
- In general, differentiation in the complex numbers corresponds to multiplication by t in the time scale

Future Works

- Investigate how various differential equations behave with δ_k
- Operator equations with $\mathscr{U}$ in quantum calculus

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